## 3. Inclusion-exclusion principle

**To read:**
[1] 2.2.1. Induction, 2.3. Inclusion-Exclusion.
[3] 3.7. Inclusion - Exclusion, 3.8. The hat-check lady.

### 3.1. **Inclusion-exclusion principle.**

**Theorem 3.1.** *(Inclusion-Exclusion principle). Let $A_1, \dots, A_n$ be finite sets. Then, the following holds*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i\cap A_j\cap A_k| - \ldots + (-1)^{n-1}|A_1\cap A_2\cap\cdots\cap A_n|.$$

*Proof.* Suppose that an element $a \in \bigcup_{i=1}^{n} A_i$ belongs to exactly $k$ different sets.
How many times did we count $a$ in the inclusion-exclusion formula

$$\sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \ldots \quad ?$$

Element $a$ is counted $(-1)^{\ell-1}\binom{k}{\ell}$ times in the $\ell$-th sum as $\ell$ goes from 1 to $n$. By the binomial theorem we have

$$\sum_{\ell=1}^{n} (-1)^{\ell-1} \binom{k}{\ell} = 1.$$

Therefore, each element $a$ is counted exactly once. This finishes the proof. $\square$

### 3.2. **Number of permutations without fixed points.**

A hat-check girl completely loses track of which of $n$ hats belong to which owners, and hands them back at random to their $n$ owners as the latter leave. What is the probability $p_n$ that nobody receives their own hat back?

This question can be reformulated in the following way: find the number of permutations of the set $\{1, 2, \dots, n\}$ without fixed points. In order to count these, we apply the inclusion-exclusion principle. Let $A$ be the set of all permutations and $A_i$ be the set of permutations of the set $\{1, 2, \dots, n\}$ for which $i$ is a fixed point. The number of permutations with no fixed points is

$$|A| - \left|\bigcup_{i=1}^{n} A_i\right|.$$

We know that $|A| = n!$, so we need to count $|\bigcup_{i=1}^{n} A_i|$. We do this using the inclusion principle. Note that $A_i \cap A_j$ represents the set of all permutations for which $i$ and $j$ are fixed points. One can see that $|A_i| = (n-1)!$ for all $i$, while $|A_i \cap A_j| = (n-2)!$. Using the same idea, we obtain $|A_i \cap A_j \cap A_k| = (n-3)!$ and so on. Altogether, this gives

$$\begin{aligned}
|A| - \left|\bigcup_{i=1}^{n} A_i\right| &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \ldots \\
&= n! - \frac{n!(n-1)!}{1!(n-1)!} + \frac{n!(n-2)!}{2!(n-2)!} - \ldots \\
&= n!(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \ldots) \\
&\approx n! \exp(-1).
\end{aligned}$$

Thus we see that the probability $p_n$ that nobody receives their own hat back is

$$p_n = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^n \frac{1}{n!}$$

As $n$ goes to infinity this number converges to $\frac{1}{e} \approx 0.37$.

3.3. **Euler's totient function.** In number theory, Euler's totient function $\phi(n)$ counts the positive integers up to a given integer $n$ that are relatively prime to $n$. For example, among the numbers $\{1, 2, 3, 4, 5, 6\}$ only 1 and 5 are coprime to 6. Therefore, we find that $\phi(6) = 2$. If $p$ is a prime number then $\phi(p) = p - 1$ and $\phi(p^k) = p^k - p^{k-1}$.

**Proposition 3.2.** *Suppose that a number $n$ has the prime factorization $n = p_1^{k_1} \cdots p_m^{k_m}$. Then by the inclusion-exclusion principle we find*

$$\phi(n) = n - \sum_{1 \le i \le m} \frac{n}{p_i} + \sum_{1 \le i < j \le m} \frac{n}{p_i p_j} - \sum_{1 \le i < j < k \le m} \frac{n}{p_i p_j p_k} + \ldots = n \prod_{i=1}^{m} (1 - \frac{1}{p_i}).$$

*Proof.* Let $A$ be the set of all numbers in $[n]$ *not coprime* with $n$.
Let $A_i$ be the set of all numbers in $[n]$ divisible by $p_i$.
Then $A = \bigcup_{i=1}^m A_i$ and $|A_i| = \frac{n}{p_i}$, $|A_i \cap A_j| = \frac{n}{p_i p_j}$, and so on. By the inclusion-exclusion formula we find

$$\begin{aligned} \phi(n) =& n - |A| \\ =& n - \sum_{1 \le i \le m} \frac{n}{p_i} + \sum_{1 \le i < j \le m} \frac{n}{p_i p_j} - \sum_{1 \le i < j < k \le m} \frac{n}{p_i p_j p_k} + \ldots = n \prod_{i=1}^{m} (1 - \frac{1}{p_i}). \end{aligned}$$

□

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## References

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